SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK

TECHNICAL REPORT NO. 346

July 1977
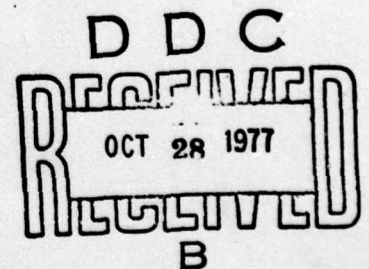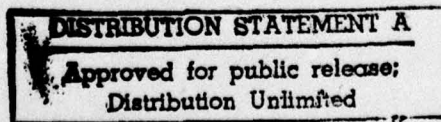

DENSE FAMILIES OF LOW-COMPLEXITY

ATTAINABLE SETS OF MARKETS

by

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Abstract

Given the attainable set of utility outcomes for a market (with finitely many traders), its complexity is defined to be the least number of commodities needed for any market giving the same set. This notion is investigated both in the case of quasiconcave and concave utility functions. It is shown that, in either case, there is a dense collection of attainable sets having complexity at most $n(n-1)/2$.

1. Introduction. The question of which n-person cooperative games can
arise from economic markets has generated much recent interest. One aspect
of this question concerns representations of "attainable sets" of markets.
In particular, representations are sought which associate with an attain-
able set a market of low "complexity" (that is, a market involving as small
a number of commodities as possible).

Two classes of markets have been considered in some detail. The
broader class consists of markets in which all of the traders' utility
functions are upper-semicontinuous and quasiconcave. We shall show that
all n-dimensional attainable sets of two recently-studied types arising
from such markets can be represented by n-trader markets involving at most
$n(n-1)/2$ commodities. A consequence of this is that there is a collection
of attainable sets, each of complexity at most $n(n-1)/2$, which is dense in
the collection of all n-dimensional attainable sets.

A natural subclass of markets consists of those in which the traders'
utility functions are continuous and concave. It is known that the
n-dimensional attainable sets of such markets are precisely the convex,
compactly generated sets in R^n. An upper bound on the complexity of these
sets, due to Kalai and Smorodinsky [3], is $(n-1)^2 - (n-2)$. In this case
also, we show that a dense collection of such n-dimensional attainable sets
has complexity bounded by $n(n-1)/2$. This has been conjectured to be the
upper bound over all n-dimensional attainable sets.

Consider a market consisting of a set of traders $N = \{1,2,\ldots,n\}$,
and an m-dimensional commodity space $I^m = \{(y_1,\ldots,y_m): 0 \leq y_i \leq 1$
for all i}. (We take $I^0 = R^0 = \{0\}$.) For any collection $\{u_i\}_{i=1}^{n}$ of
utility functions of the traders (real-valued functions on I^m), the
attainable set of the market is

$$A(u_1,\ldots,u_n) = \{x \epsilon R^n: x \leq (u_1(y^1),\ldots,u_n(y^n)), \text{ where}$$

$$\text{each } y^i \epsilon I^m \text{ and } \sum y^i = (1,\ldots,1)\}.$$

This is the set of all utility outcomes which can be achieved by some distribution of the available commodities among the traders.

A set X in R^n is the <u>comprehensive hull</u> of another set Y if $X = \{x \epsilon R^n: x \leq y \text{ for some } y \epsilon Y\}$; in this case, we say that X is (comprehensively) <u>generated by</u> Y. It is not difficult to show that if $u_1,\ldots,u_n$ are upper-semicontinuous and bounded, then $A(u_1,\ldots,u_n)$ is compactly generated (generated by a compact set).

Let U_1 be the collection of all upper-semicontinuous, quasiconcave utility functions, and let U_2 be the subcollection of all continuous, concave utility functions. For $k = 1,2$, let $A_k(n)$ be the collection of all n-dimensional attainable sets of markets in which the traders' utility functions are in U_k. The extent of $A_1(n)$ was investigated in [4], [5], [6] and [7]; in [1], the sets in $A_2(n)$ were characterized as all sets which are generated by convex, compact sets.

Let V be an attainable set in $A_k(n)$. If $u_1,\ldots,u_n$ are functions in U_k defined on I^m (for some fixed m), such that $V = A(u_1,\ldots,u_n)$, then $\{u_i\}_{i=1}^n$ will be called a k-representation for V over I^m. The <u>k-complexity</u> of V is the least $m \geq 0$ such that there exists a k-representation for V over I^m. We use comp V to denote the k-complexity of V; the context will make it clear whether k is 1 or 2.

2. <u>Quasiconcave markets</u>. In this section, we consider the attainable sets of markets in which all traders' utility functions are upper-semicontinuous and quasiconcave. Therefore, we direct our attention to 1-representations and 1-complexity. Without loss of generality, we assume that all n-dimensional attainable sets under consideration are comprehensively generated by compact sets lying in the interior of the unit n-cube I^n. Let D_k be the "corner" generated by the point $(1,\ldots,1,0) \epsilon R^k$. In [6] and [7], n-dimensional attainable sets were represented by utility functions obtained from constructions which first represented (over I^{n-1}) the unions of these sets with D_n. This makes the following result of value.

<u>Theorem 2.1</u>: Let V be generated by a nonempty compact set in I^n. For $1 \le k \le n$, let $V'_k = \{x \epsilon R^k : (x;0) \epsilon V\}$ and let $V_k = V'_k \cup D_k$. If each V_k is an attainable set (in $A_1(k)$), then V is an attainable set (in $A_1(n)$), and comp $V \le \sum_{k=1}^{n}$ comp V_k.

<u>Proof</u>: Let $c_k = $ comp V_k, and $c = \sum c_k$. Assume that $V_k = A(u_1^k,\ldots,u_k^k)$, where all u_i^k $(1 \le i \le k)$ are defined on I^{c_k}. Represent any $x \epsilon I^c$ as $x = (^1x,\ldots,^nx)$, where $^kx \epsilon I^{c_k}$. Define $u_i(x) = \min(u_i^i(^ix),\ldots,u_i^n(^nx))$. We shall verify that $V = A(u_1,\ldots,u_n)$, from which the upper bound on the complexity of V follows immediately.

Consider any $z = (z_1,\ldots,z_n) \epsilon A(u_1,\ldots,u_n)$. There is some allocation $(x^1,\ldots,x^n)$, with each $x^i \epsilon I^c$ and $\sum x^i = (1,\ldots,1)$, such that $u_i(x^i) \ge z_i$ for all $1 \le i \le n$. Therefore, by definition, $u_i^j(^jx^i) \ge z_i$ for all $j \ge i$. Hence, $z^{(k)} = (z_1,\ldots,z_k) \epsilon V_k$ for all $1 \le k \le n$. If $z^{(k)} \notin V'_k$, then $z^{(k)} \le (1,\ldots,1,0)$ and therefore $z_k \le 0$. Let $\overline{k} = \max(0, \{k: z^{(k)} \epsilon V'_k\})$. If $\overline{k} = 0$, then $z \le (0,\ldots,0) \epsilon V$. If $\overline{k} > 0$, then $z^{(\overline{k})} \epsilon V'_{\overline{k}}$ and $z \le (z^{(\overline{k})};0) \epsilon V$. In either case, we conclude that $z \epsilon V$ (and, indeed, $\overline{k} = n$) and therefore $V \supset A(u_1,\ldots,u_n)$.

On the other hand, given any $z \epsilon V$, it follows that each $z^{(k)} =$ $(z_1, \ldots, z_k) \epsilon V_k' \subset V_k$. For each $1 \leq k \leq n$, let $(^kx^1, \ldots, ^kx^k)$ be an allocation, with each $^kx^i \epsilon I^{c_k}$ and $\sum_i {}^kx^i = (1, \ldots, 1)$, such that $z^{(k)} \leq (u_1^k(^kx^1), \ldots, u_k^k(^kx^k)) \epsilon V_k$. Define $x^i = (0; {}^ix^i, \ldots, {}^nx^i) \epsilon I^c$ for each $1 \leq i \leq n$. Then $z \leq (u_1(x^1), \ldots, u_n(x^n))$, and therefore $V \subset A(u_1, \ldots, u_n)$.

Finally, it may be observed that the construction of each u_i from the u_i^k ($k \geq i$) preserves both upper-semicontinuity and quasiconcavity. This completes the proof of the theorem.

A set $V \subset R^n$ is <u>finitely generated</u> if it is the comprehensive hull of a finite set (that is, if V is the union of a finite number of corners); V is <u>convexifiable</u> if it is compactly generated and there exists a collection $\{g_1, \ldots, g_n\}$ of strictly increasing, continuous functions such that

$$V_{\{g_1, \ldots, g_n\}} = \{x \epsilon R^n : x \leq (g_1(y_1), \ldots, g_n(y_n)) \text{ for some } y \epsilon V\}$$

is convex.

Let V be generated by a set in I^n. It has been shown that if V is finitely generated (see [6]) or convexifiable (see [7]), then V_n is an attainable set of complexity at most $n-1$. Since these properties, of being finitely generated or of being convexifiable, are inherited by all V_k' ($1 \leq k \leq n$), an application of the theorem yields the following result.

<u>Corollary 2.2</u>: If $V \subset R^n$ is finitely generated or convexifiable, then V is an attainable set (in $A_1(n)$), and comp $V \leq n(n-1)/2$.

Since the finitely generated sets are (Hausdorff) dense in $A_1(n)$ ([2, Theorem 2]), we can state the following.

<u>Corollary 2.3</u>: $A_1(n)$ has a dense subset consisting of attainable sets of (1-) complexity no greater than $n(n-1)/2$.

3. <u>Concave markets</u>. We now turn our attention to the attainable sets of markets in which all traders' utility functions are continuous and concave, and in consequence we consider 2-representations and 2-complexity. Without loss of generality, we continue to assume that all n-dimensional attainable sets under consideration are generated by (convex, compact) sets lying in the interior of I^n. For any $h \geq 0$, let $D_k(h)$ be the corner generated by the point $(1,\ldots,1,-h) \epsilon R^k$. In [1], n-dimensional attainable sets were represented by utility functions obtained from contructions which first represented (over I^{n-1}) the convex hulls of the unions of these sets with sets $D_n(h)$, for large values of h. Therefore, a variation of the theorem of the previous section will be broadly applicable.

Let V be generated by a convex compact set in I^n, and, as before, let $V'_k = \{x \epsilon R^k: (x;0) \epsilon V\}$. For $2 \leq k \leq n$, let $V'_{k-1} \times R^1 = \{x \epsilon R^k: (x_1,\ldots,x_{k-1}) \epsilon V'_{k-1}\}$ be a cylinder with cross-section V'_{k-1}. We define $V'_0 = \{0\}$ and $V'_0 \times R^1 = R^1$. For any $h \geq 0$, define $V_k(h)$ to be the (comprehensive) convex hull of $V'_k \cup D_k(h)$. Each $V_k(h)$ is generated by a convex compact set, and is therefore an attainable set (in $A_2(k)$). We say V is <u>resolved</u> by a sequence of non-negative numbers $h_1,\ldots,h_n$ if for all $1 \leq k \leq n$, $V_k(h_k) \cap (V'_{k-1} \times R^1) = V'_k$; in this case, we say V is <u>resolvable</u>.

<u>Theorem 3.1</u>: Let V be generated by a convex compact set in I^n. Assume that V is resolved by $h_1,\ldots,h_n$. Then $\text{comp } V \leq \sum_{k=1}^{n} \text{comp } V_k(h_k) \leq n(n-1)/2$.

<u>Proof</u>. We begin as in the proof of Theorem 2.1. Let $c_k = \text{comp } V_k(h_k)$, and $c = \sum c_k$. Assume that $V_k(h_k) = A(u_1^k,\ldots,u_k^k)$, where all u_i^k $(1 \leq i \leq k)$ are continuous concave functions defined on I^{c_k}. Write $x \epsilon I^c$ as

$x = (^1x, \ldots, ^nx)$, where $^k x \in I^{c_k}$. Define $u_i(x) = \min(u_i^i(^ix), \ldots, u_i^n(^nx))$. The construction of the functions $u_1, \ldots, u_n$ yields continuous concave functions on I^c. Therefore, it will suffice to verify that $V = A(u_1, \ldots, u_n)$.

Consider any $z = (z_1, \ldots, z_n) \in A(u_1, \ldots, u_n)$. Then each $z^{(k)} = (z_1, \ldots, z_k) \in V_k(h_k)$. Assume that, for some $2 \leq k \leq n$, $z^{(k)} \notin V_k'$. Since V is resolved by $h_1, \ldots, h_n$, $V_k(h_k) \cap (V_{k-1}' \times R^1) = V_k'$; therefore $z^{(k)} \notin V_{k-1}' \times R^1$, and hence $z^{(k-1)} \notin V_{k-1}'$. Now, assume that $z \notin V = V_n'$. From the result just established, it follows inductively that $z^{(k)} \notin V_k'$ for all $1 \leq k \leq n$. But $z^{(1)} \in V_1(h_1) = V_1'$. This contradiction affirms that $z \in V$. Therefore, $V \supset A(u_1, \ldots, u_n)$.

On the other hand, it follows as in the proof of Theorem 2.1 that $V \subset A(u_1, \ldots, u_n)$. Therefore $V = A(u_1, \ldots, u_n)$, and the complexity of V is no greater than $c = \sum_{k=1}^{n} \text{comp } V_k(h_k)$. The second inequality follows from the observation that for any $h \geq 0$, $\text{comp } V_k(h) \leq (k-1)$; see [1].

In the remainder of this section, we will show that there is a dense (in the sense of the Hausdorff distance) collection of resolvable n-dimensional attainable sets. This class consists of those attainable sets having a (uniformly) positive normal at every nonnegative boundary points; these will be referred to as positively supported sets. Specifically, an n-dimensional attainable set V is said to be (uniformly) positively supported if there exists a closed set $Q \subset \{q \in R^n \mid \sum_{i=1}^{m} q_i = 1, q > 0\}$ so that for each $x \in \partial V \cap R_+^n$, there is a $q \in Q$ for which $\langle x, q \rangle \geq \langle y, q \rangle$ for all $y \in V$. Here ∂V denotes the boundary of V, R_+^n, the nonnegative orthant in R^n, and $\langle \cdot, \cdot \rangle$, the usual inner product on R^n.)

Lemma 3.2: If V is positively supported then V is resolvable.

<u>Proof</u>: Let Q be the set of normals specified in the definition. Let $h_1 = 0$ and, for $2 \le k \le n$, let $h_k = \max \{ \sum_{i<k} q_i/q_k | q \epsilon Q \} > 0$. We shall show that V is resolved by the sequence $h_1, h_2, \ldots, h_n$. Since, for $1 \le k \le n$, we always have

$$V_k(h_k) \cap (V'_{k-1} \times R^1) \supset V'_k \qquad (3.2.1)$$

and, for $k = 1$, equality is clear, it is enough to show the other inclusion for $k \ge 2$.

Suppose $x \epsilon R^k$ is an element of the left-hand side of (3.2.1). If $x_k \le 0$, then since $(x_1, \ldots, x_{k-1}) \epsilon V'_{k-1}$, we must have $(x_1, \ldots, x_{k-1}, 0) \epsilon V'_k$ and so $x \epsilon V'_k$. If $x_k > 0$, then since $x \epsilon V_k(h_k)$, we must have $x \le \bar{x}$ where

$$0 \le \bar{x} = \lambda y + (1-\lambda)(1,1,\ldots,1,-h_k) \epsilon V_k(h_k), \qquad (3.2.2)$$

$y \epsilon V'_k \cap R^k_+$ and $0 < \lambda \le 1$. We show $\bar{x} \epsilon V'_k$, and so $x \epsilon V'_k$ by comprehensiveness. If $\bar{x} \not\epsilon V'_k$ then there exists $v \epsilon \partial V'_k$, $0 \le v \le \bar{x}$, $v \ne \bar{x}$. Since $(v;0) \epsilon \partial V \cap R^n_+$, there is a $q \epsilon Q$ so that

$$\langle \bar{x}, q^{(k)} \rangle > \langle v, q^{(k)} \rangle \ge \langle z, q^{(k)} \rangle \qquad (3.2.3)$$

for all $z \epsilon V'_k$, where $q^{(k)} = (q_1, \ldots, q_k) > 0$. Further,

$$\langle v, q^{(k)} \rangle \ge 0 \ge \sum_{i<k} q_i - h_k q_k = \langle (1,1,\ldots,1,-h_k), q^{(k)} \rangle$$

so $\langle v, q^{(k)} \rangle \ge \langle z, q^{(k)} \rangle$ for all $z \epsilon V_k(h_k)$. This contradicts (3.2.2) and

(3.2.2), establishing $\bar{x} \epsilon V'_k$. In either case, we have $x \epsilon V'_k$ and thus have shown (3.2.1) to be an equality, completing the proof.

It seems intuitively clear that the positively supported attainable sets are dense among all such sets. We outline a proof here for completeness. We denote by d the Hausdorff distance induced by the norm

$$\| \, x \, \| = \max_{1 \le i \le m} |x_i|.$$

Suppose V is an n-dimensional attainable set (generated by a compact convex subset of I^n). We can write

$$V = \{x \epsilon R^n | \ \langle x,p \rangle \le \alpha_p \ \text{ for } \ p \epsilon S\}$$

where $S = \{p \epsilon R^n_+ | \ \sum_{i=1}^{n} p_i = 1\}$ and $\alpha_p = \sup_{x \epsilon V} \langle x,p \rangle$. Note that α_p, the support function of V, varies continuously for $p \epsilon S$. For $0 < \epsilon < 1/n$, define

$$V^\epsilon = \{x \epsilon R^n_+ | \ \langle x,p \rangle \le \alpha_p \ \text{ for } \ p \epsilon Q^\epsilon\} - R^n_+$$

where $Q^\epsilon = \{q \epsilon S | q_i \ge \epsilon \ \text{ for all } \ i\}$.

Lemma 3.3: The sets V^ϵ are positively supported and $d(V,V^\epsilon) \to 0$ as $\epsilon \to 0$.

Proof: V^ϵ is positively supported since for $x \epsilon \partial V^\epsilon \cap R^n_+$, there is a $p \epsilon Q^\epsilon$ such that $\langle x,p \rangle = \alpha_p \ge \langle y,p \rangle$ for all $y \epsilon V^\epsilon$ (otherwise x would be in the interior of V^ϵ).

Since $d(V,V^\epsilon) \le d(V \cap R^n_+, \ V^\epsilon \cap R^n_+)$ and $V^\epsilon \supset V$, it is enough show that, given $\eta > 0$, there is $\bar{\epsilon} > 0$ so that for $\epsilon < \bar{\epsilon}$ and

$x \epsilon V^{\epsilon} \backslash V$, $x \geq 0$, there is $y \epsilon V$, $y \geq 0$ so that $||x - y|| \leq \eta$.

Let $\epsilon^{o} \sim 1/2n$. The function $f(z,p) = \langle z,p \rangle - \alpha_{p}$ is uniformly continuous over $(V^{\epsilon^{o}} \cap R_{+}^{n}) \times S$ and so there is an $\overline{\epsilon} < \epsilon^{o}$ so that for $z \epsilon V^{\epsilon^{o}}$, $z \geq 0$, and $p,q \epsilon S$, we have $||q - p|| \leq \overline{\epsilon}$ implies $|f(z,q) - f(z,p)| \leq \eta$.

Take $\epsilon < \overline{\epsilon}$ and $x \epsilon V^{\epsilon} \backslash V$, $x \geq 0$. Let $q \epsilon S$ maximize $f(x,p)$ over S. Since $x \epsilon V^{\epsilon} \backslash V$, $\delta = f(x,q) > 0$ but $f(x,p) \leq 0$ for $p \epsilon Q^{\epsilon}$. Now let $z = x - (\delta,\delta,\ldots,\delta)$. Then $z \epsilon V$ and $y = z^{+} \epsilon V \cap R_{+}^{n}$ where $(z^{+})_{i} = \max(z_{i},0)$. Further $z \leq y \leq x$, so

$$||x - y|| \leq ||x - z|| = \delta \leq \eta,$$

proving the lemma.

Combining (3.2) and (3.3), we have our final result.

Theorem 3.4: $A_{2}(n)$ has a dense subset consisting of attainable sets of (2-) complexity no greater than $n(n-1)/2$.

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Technical Report No. 346		TR-346

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
DENSE FAMILIES OF LOW-COMPLEXITY ATTAINABLE SETS OF MARKETS	Technical Report
	6. PERFORMING ORG. REPORT NUMBER
	T.R. No. 346

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Louis J. Billera and Robert J. Weber	N00014-75-C-0678, NSF-MCS-75-02024

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
School of Operations Research and Industrial Engineering, Cornell University, Upson Hall Ithaca, New York 14853	

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
Operations Research Program Office of Naval Research Arlington, VA 22217	July 1977
	13. NUMBER OF PAGES
	10 14p.

11. CONTROLLING OFFICE NAME AND ADDRESS	15. SECURITY CLASS. (of this report)
	Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Unlimited

18. SUPPLEMENTARY NOTES

409 869

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Attainable Sets   Concave Markets
Pareto Sets   Quasiconcave Markets
Trading Economy   Utility Outcomes
Complexity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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